Deformed Fokker-Planck Equations

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Based on the well-known relation between Fokker-Planck equations and Schrödinger equations of quantum mechanics (QM), we propose new *deformed* Fokker-Planck (FP) equations associated with the Schrödinger equations of "discrete" QM. The latter is a natural discretization of QM and its Schrödinger equations are difference instead of differential equations. Exactly solvable FP equations are obtained corresponding to exactly solvable "discrete" QM, whose eigenfunctions include various deformations of the classical orthogonal polynomials.

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1. Since it was first introduced by Fokker and Planck to describe the Brownian motion of particles, the Fokker-Planck (FP) equation has become one of the basic tools used to deal with fluctuations in various kinds of systems [1]. Recently, the phenomena of anomalous diffusion in fractal and disordered media have prompted new developments in the FP theory. Most attempts have been along developing fractional FP equations, where the ordinary spatial derivatives are replaced by fractional derivatives [2]. Others have tried to generalize the linear FP equation to nonlinear ones [3]. Diffusion equation based on q-derivatives has also been considered [4].

In this paper, we shall generalize the FP equation in a different direction. We shall derive new types of deformed FP equation which are associated with the Schrödinger equations of the "discrete" quantum mechanics considered in [5, 6]. The eigenfunctions of some exactly solvable "discrete" quantum mechanics include various deformations of the classical orthogonal polynomials (the Hermite, Laguerre and Jacobi polynomials), namely, those belonging to the family of Askey-scheme of hypergeometric orthogonal polynomials, and the Askey-Wilson polynomial in q-analysis [7, 8, 9]. The discrete Schrödinger equation is a discrete deformation of the usual Schrödinger equation in much the same way that the Ruijsenaars-Schneider-van Diejen type systems [10] are the "discrete" counterparts of the Calogero and Sutherland systems [11], the celebrated exactly solvable multi-particle dynamics. In fact, these deformed orthogonal polynomials arise in the problem of describing the equilibrium positions of Ruijsenaars-Schneider type systems [6], corresponding to the facts that the Hermite, Laguerre and Jacobi polynomials describe the equilibrium positions of the Calogero and Sutherland systems [12]. In this sense, our new deformed FP equations can be considered as the "discrete" deformation of the usual FP equation.

2. In one dimension, the FP equation of the probability density $P(x,t)$ is [1]

$$\frac{\partial P(x,t)}{\partial t} = L_{FP} P(x,t),$$
$$L_{FP} \equiv -\frac{\partial}{\partial x} D^{(1)}(x) + \frac{\partial^2}{\partial x^2} D^{(2)}(x). \tag{1}$$

The functions $D^{(1)}(x)$ and $D^{(2)}(x)$ in the FP operator L_{FP} are, respectively, the drift and the diffusion coefficient (we consider only time-independent case). The drift coefficient represents the external force acting on the particle, while the diffusion coefficient accounts for the effect of fluctuation. The drift coefficient is usually expressed in terms of a drift potential $\Phi(x)$ according to $D^{(1)}(x) = -\Phi'(x)$, where the prime denotes derivative with respect to x.

It is well known that the FP equation is closely related to the Schrödinger equation. The correspondence between the two equations is usually given by transforming the FP equation into the corresponding Schrödinger equation [1]. But the argument could well be reversed.

Consider a quantum mechanical Hamiltonian of one degree of freedom (we adopt the unit system in which $\hbar$ and the mass m of the particle are such that $\hbar = 2m = 1$)

$$H = -\frac{\partial^2}{\partial x^2} + V(x). \tag{2}$$

Let $V(x)$ be such that the ground state energy is zero, i.e. $H\phi_0 = 0$. The Hamiltonian H is completely determined by its ground state wave function $\phi_0(x)$. By the well-known theorem of quantum mechanics, $\phi_0(x)$ has no node and can be chosen real. Thus it can be parametrized by a real *prepotential* $W(x)$:

$$\phi_0(x) = e^{-W(x)}, \tag{3}$$
$$V(x) = W'(x)^2 - W''(x). \tag{4}$$

The Hamiltonian is factorizable and positive semi-definite: $H = A^\dagger A$. Here $A^\dagger \equiv -\partial_x + W'$ and $A \equiv \partial_x + W'$. The constant part of $W(x)$ is so chosen as to normalize $\phi_0(x)$ properly, $\int \phi_0(x)^2\, dx = 1$.

Now we define an operator from the H and ϕ_0 by the similarity transformation

$$L_{FP} \equiv -\phi_0 H \phi_0^{-1}\ , \tag{5}$$

which guarantees the non-positivity of the eigenvalues of L_{FP}. With ϕ_0 given in (3), one obtains

$$L_{FP} = \frac{\partial}{\partial x} 2W' + \frac{\partial^2}{\partial x^2}. \tag{6}$$

From (1), it is seen that L_{FP} is just the corresponding FP operator with $\Phi(x) = 2W$ as the drift potential and with a constant diffusion coefficient (here equals one owing to the choice of unit in H). Hence both H and L_{FP} are determined by ϕ_0. The eigenfunction $P_n(x)$ of L_{FP} corresponding to eigenvalue $-\lambda_n$ is related to the (real and normalized) eigenfunction ϕ_n of H corresponding to λ_n by $P_n(x) = \phi_0(x)\phi_n(x)$. The stationary distribution is $P_0 = \phi_0^2 = \exp(-2W)$, which is obviously non-negative, and is the zero mode of L_{FP}: $L_{FP}P_0 = 0$. Any positive definite initial probability density $P(x, 0)$ can be expanded as $P(x, 0) = \phi_0(x) \sum_n c_n \phi_n(x)$, with constant coefficients c_n $(n = 0, 1, \ldots)$

$$c_n = \int_{-\infty}^{\infty} \phi_n(x) \left(\phi_0^{-1}(x) P(x, 0)\right) dx. \tag{7}$$

Then at any later time t, the solution of the FP equation is $P(x, t) = \phi_0(x) \sum_n c_n \phi_n(x) \exp(-\lambda_n t)$.

3. We now derive a new class of deformed FP equation corresponding to the discrete Schrödinger equations discussed in [5] in accordance with the prescription in the last section. Eigenfunctions in this type of discrete Schrödinger equations are related to the family of Askey-scheme of hypergeometric orthogonal polynomials [7, 8].

The Hamiltonian has the general form

$$\begin{aligned} H \equiv &\sqrt{V(x)}\, e^{-i\partial_x} \sqrt{V^*(x)} + \sqrt{V^*(x)}\, e^{i\partial_x} \sqrt{V(x)} \\ &- \left(V(x) + V^*(x)\right). \end{aligned} \tag{8}$$

Here the momentum operator $p = -i\partial_x$ (with $\hbar = 1$) appears as exponentiated instead of powers in ordinary quantum mechanics. Thus they cause a finite shift of the wave function in the *imaginary* direction: $e^{\pm i\partial_x}\phi(x) = \phi(x \pm i)$. Throughout this paper the following convention of a complex conjugate function will be adopted: for an arbitrary function $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$, we define $f^*(x) = \sum_n a_n^* x^n$. Here a_n^* is the complex conjugation of a_n. Note that $f^*(x)$ is not the complex conjugation of $f(x)$, $(f(x))^* = f^*(x^*)$. This is particularly important when a function is shifted in the imaginary direction.

The Hamiltonian (8) is factorised, i.e. $H = A^\dagger A$ with

$$A \equiv e^{-\frac{i}{2}\partial_x} \sqrt{V^*(x)} - e^{\frac{i}{2}\partial_x} \sqrt{V(x)}, \tag{9}$$

$$A^\dagger \equiv \sqrt{V(x)}\, e^{-\frac{i}{2}\partial_x} - \sqrt{V^*(x)}\, e^{\frac{i}{2}\partial_x}. \tag{10}$$

Here $\dagger$ denotes the ordinary hermitian conjugation with respect to the ordinary L^2 inner product: $\langle f|g\rangle = \int_{-\infty}^{\infty} (f(x))^* g(x) dx$. Obviously the Hamiltonian (8) is hermitian (self-conjugate) and positive semi-definite.

Let the ground state ϕ_0 be annihilated by A:

$$A\phi_0(x) = 0, \quad (\Longrightarrow H\phi_0(x) = 0, \quad \lambda_0 = 0). \tag{11}$$

Explicitly the above equation reads

$$\sqrt{V^*\left(x - i/2\right)}\; \phi_0\left(x - i/2\right) = \sqrt{V\left(x + i/2\right)}\; \phi_0\left(x + i/2\right), \tag{12}$$

or equivalently

$$\sqrt{V^*(x - i)}\; \phi_0(x - i) = \sqrt{V(x)}\; \phi_0(x), \tag{13}$$

$$\sqrt{V(x + i)}\; \phi_0(x + i) = \sqrt{V^*(x)}\; \phi_0(x). \tag{14}$$

Eq. (12) relates the potential $V(x)$ and the ground state ϕ_0, and hence is the discrete analogue of (3) albeit in an implicit way.

Now, we form the associated FP operator from (8) and ϕ_0 in (11) according to the similarity transformation (5). The sum $V(x) + V^*(x)$ remains intact. The first term in H transforms as

$$\begin{aligned} &\phi_0(x) \sqrt{V(x)}\, e^{-i\partial_x} \sqrt{V^*(x)}\, \phi_0^{-1}(x) \\ = &\left(\frac{\phi_0(x)\sqrt{V(x)}}{\phi_0(x - i)\sqrt{V^*(x - i)}} \right) e^{-i\partial_x} V^*(x). \end{aligned} \tag{15}$$

By virtue of (13), the term in the bracket of the r.h.s. of (15) is simply unity. Hence

$$\phi_0(x) \sqrt{V(x)}\, e^{-i\partial_x} \sqrt{V^*(x)}\, \phi_0^{-1}(x) = e^{-i\partial_x} V^*(x). \tag{16}$$

Similarly, using (14), the second term in H transforms as

$$\phi_0(x) \sqrt{V^*(x)}\, e^{i\partial_x} \sqrt{V(x)}\, \phi_0^{-1}(x) = e^{i\partial_x} V(x). \tag{17}$$

Putting everything together, the result is

$$L_{FP} = -e^{i\partial_x} V(x) - e^{-i\partial_x} V^*(x) + V(x) + V^*(x). \tag{18}$$

This is the general form of FP operator corresponding to the discrete Hamiltonian H in (8). One has $L_{FP}\phi_0^2 = 0$ as a consequence of $H\phi_0 = 0$. Thus ϕ_0^2 is the stationary solution of the respective FP equation.

We now discuss the limiting form of the FP equation when the momentum $p = -i\partial_x$ is small. We expand the operators $e^{\pm i\partial_x}$ in (18) up to the 2nd order in p. The FP operator becomes

$$L_{FP} = \frac{\partial}{\partial x} 2Im\ V(x) + \frac{\partial^2}{\partial x^2} 2Re\ V(x). \tag{19}$$

Hence, in this limit, the deformed FP equation (18) does reduce to the usual FP equation (1), with $D^{(1)} = -2Im\ V(x)$ and $D^{(2)} = 2Re\ V(x)$.

Let us illustrate this connection with a simple example discussed in [5], namely, the discrete Schrödinger equation with the Meixner-Pollaczek polynomials as eigenfunctions. These polynomials are deformation of the Hermite polynomials. The potential in this system is $V(x) = a + ix$ with a real and positive. From the above discussion, the small momentum limit of the FP equation is just the ordinary FP equation with $D^{(1)} = -2x$ and $D^{(2)} = 2a$. This latter system is none other than the FP equation of the celebrated Ornstein-Uhlenbeck process,

which can be exactly solved by means of expansion in terms of the Hermite polynomials [1].

4. In [5] four shape-invariant potentials [13] in the discrete quantum mechanics were presented, including the Meixner-Pollaczek case discussed above. Shape invariance ensures that the corresponding Schrödinger equations are exactly solvable. Hence, the corresponding deformed FP equations are also exactly solvable. For completeness, we list below the potentials $V(x)$, the corresponding ground state wave functions $\phi_0(x)$, and the eigenvalues λ_n $(n=0,1,2,\ldots)$. Needless to say all the eigenfunctions are square integrable. All parameters a, b, c and d are assumed to be real and positive. We name the case by the type of polynomials to which the polynomial part of ϕ_n belongs (see [5] for the details of these polynomials).

(i) the Meixner-Pollaczek case,
(one-parameter deformation of the Hermite polynomial)

$$\begin{aligned} V(x) &= a+ix, \\ \phi_0(x) &\propto |\Gamma(a+ix)|, \\ \lambda_n &= 2n; \end{aligned}$$

(ii) the continuous Hahn case,
(two-parameter deformation of the Hermite polynomial)

$$\begin{aligned} V(x) &= (a+ix)(b+ix), \\ \phi_0(x) &\propto |\Gamma(a+ix)\Gamma(b+ix)|, \\ \lambda_n &= n(n+2a+2b-1); \end{aligned}$$

(iii) the continuous dual Hahn case,
(two-parameter deformation of the Laguerre polynomial)

$$\begin{aligned} V(x) &= \frac{(a+ix)(b+ix)(c+ix)}{2ix(2ix+1)}, \\ \phi_0(x) &\propto \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|, \\ \lambda_n &= n; \end{aligned}$$

(iv) the Wilson case,
(three-parameter deformation of the Laguerre polynomial)

$$\begin{aligned} V(x) &= \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)}, \\ \phi_0(x) &\propto \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|, \\ \lambda_n &= n(n+a+b+c+d-1). \end{aligned}$$

For compactness, we have written the ground state wave function in terms of the absolute value symbol. For instance,

$$|\Gamma(a+ix)| = \sqrt{\Gamma(a+ix)\Gamma(a-ix)}. \tag{20}$$

5. The potentials listed in the last section are exactly solvable as they are shape-invariant. However, the construction of the deformed FP operator can be carried over to the more general potential

$$V(x) = \frac{\prod_j (a_j+ix)}{\prod_k (d_k+ix)}, \tag{21}$$

where a_j, d_k are real and positive, and the degree of the numerator is at least one greater than that of the denominator for square integrability. The corresponding ground state wave function is

$$\phi_0(x) \propto \sqrt{\frac{\prod_j \Gamma(a_j+ix)\Gamma(a_j-ix)}{\prod_k \Gamma(d_k+ix)\Gamma(d_k-ix)}}. \tag{22}$$

The potential $V(x)$ and the state $\phi_0(x)$ satisfy the relation (12). As discussed previously, this ensures that the Hamiltonian H with $V(x)$ in (21) is mapped into L_{FP} in (18) with the similarity transformation (5), and that the probability density ϕ_0^2 is indeed the zero mode of L_{FP}.

6. We turn now to a different type of shape-invariant "discrete" quantum mechanical single particle systems with a q-shift type kinetic term discussed in [6]. Eigenfunctions in this class of system are related to the Askey-Wilson polynomials [9]. We now derive the associated FP operator for such systems.

Following [6], we use variables θ, x and z, which are related as

$$0 \le \theta \le \pi, \quad x=\cos\theta, \quad z=e^{i\theta}. \tag{23}$$

The dynamical variable is θ and the inner product is $\langle f|g\rangle = \int_0^\pi d\theta f(\theta)^* g(\theta)$. We denote $D \equiv z\frac{d}{dz}$. Then q^D is a q-shift operator, $q^D f(z) = f(qz)$. We note here that

$$\begin{aligned} \int_0^\pi d\theta &= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}, \\ -i\frac{d}{d\theta} &= z\frac{d}{dz} = D, \quad f(z)^* = f^*(z^{-1}). \end{aligned} \tag{24}$$

For a function $V(z)$, which is a function of a real constant q $(0<q<1)$ and a set of real parameters, we define the following Hamiltonian H,

$$\begin{aligned} H \;\equiv\; & \sqrt{V(z)}\,q^D\sqrt{V^*(z^{-1})} + \sqrt{V^*(z^{-1})}\,q^{-D}\sqrt{V(z)} \\ & -\left(V(z)+V^*(z^{-1})\right). \end{aligned} \tag{25}$$

The eigenvalue equation reads $H\phi_n = \lambda_n\phi_n$ with eigenfunctions $\phi_n(z)$ and eigenvalues λ_n $(n=0,1,\ldots)$ (we assume non-degeneracy $\lambda_0<\lambda_1<\cdots$). The kinetic term causes a q-shift in the variable z. This Hamiltonian is factorized, i.e. $H=A^\dagger A$, with

$$A = q^{\frac{D}{2}}\sqrt{V^*(z^{-1})} - q^{-\frac{D}{2}}\sqrt{V(z)}, \tag{26}$$

$$A^\dagger = \sqrt{V(z)}\,q^{\frac{D}{2}} - \sqrt{V^*(z^{-1})}\,q^{-\frac{D}{2}}. \tag{27}$$

The ground state ϕ_0 is the function annihilated by A:

$$A\phi_0 = 0 \quad (\Rightarrow H\phi_0=0,\ \lambda_0=0). \tag{28}$$

Explicitly this equation reads

$$\sqrt{V^*(q^{-\frac{1}{2}}z^{-1})}\,\phi_0(q^{\frac{1}{2}}z) = \sqrt{V(q^{-\frac{1}{2}}z)}\,\phi_0(q^{-\frac{1}{2}}z). \quad (29)$$

Note that Eq. (29) implies

$$\sqrt{V^*(q^{-1}z^{-1})}\ \phi_0(qz) = \sqrt{V(z)}\ \phi_0(z), \quad (30)$$

$$\sqrt{V^*(z^{-1})}\ \phi_0(z) = \sqrt{V(q^{-1}z)}\ \phi_0(q^{-1}z). \quad (31)$$

As with (12), (29) is the analogue of (3). The other eigenfunctions can be obtained in the form

$$\phi_n(z) \propto p_n(z)\phi_0(z), \quad (32)$$

where $p_n(z)$ is a Laurent polynomial in z [6].

Using H, ϕ_0, (30) and (31), the similarity transformation (5) produces the FP operator

$$\begin{aligned} L_{FP} &= -q^{-D}V(z) - q^{D}V^*(z^{-1}) \\ &\quad +V(z) + V^*(z^{-1}). \end{aligned} \quad (33)$$

This is the general discrete q-deformed FP operator corresponding to the Hamiltonian (25). Again, L_{FP} annihilates ϕ_0^2.

As an example, let us take $V(x)$ to be the one discussed in [6]:

$$V(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}. \quad (34)$$

For simplicity we assume $-1 < a, b, c, d < 1$. Note that $V^*(z) = V(z)$. The ground state is given by [8]

$$\begin{aligned} \phi_0(z) &\propto \left|\frac{(z^2;q)_\infty}{(az,bz,cz,dz;q)_\infty}\right| \quad (35) \\ &= \sqrt{\frac{(z^2,z^{-2};q)_\infty}{(az,az^{-1},bz,bz^{-1},cz,cz^{-1},dz,dz^{-1};q)_\infty}}, \end{aligned}$$

where $(a_1,\cdots,a_m;q)_\infty = \prod_{j=1}^m\prod_{n=0}^\infty(1-a_jq^n)$. Excited states have the form (32) $\phi_n(z) \propto p_n(z)\phi_0(z)$, where $p_n(z)$ is proportional to the Askey-Wilson polynomial [8, 9], which is a three-parameter deformation of the Jacobi polynomial. The eigen-energies are

$$\lambda_n = q^{-n}(1-q^n)(1-abcdq^{n-1}). \quad (36)$$

Being shape-invariant, the Schrödinger equation of this system is exactly solvable, and so is the corresponding FP equation.

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